ON EQUILIBRIUM MODES OF A RUBBER SPHERICAL SHELL UNDER INTERNAL PRESSURE

PMM Vol. 32, No. 2, 1968, pp. 339-344

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(Received December 20, 1967)

The problem mentioned in the title has repeatedly been posed [1 to 4]. It is usually assumed that the shell retains its spherical shape during deformation. Below we consider conditions for the formation of equilibrium modes different, but similar to, the spherical one. It is assumed that the shell received a large uniform elongation in the subcritical state, and that deviations from this state are small.

1. We take the expression of the strain law for rubber as [4]

$$\begin{aligned}\sigma_1 &= \frac{\partial W}{\partial I_1}\lambda_1^2 + \frac{\partial W}{\partial I_2'}\lambda_1^2(\lambda_2^2+\lambda_3^2)+s\\ \sigma_2 &= \frac{\partial W}{\partial I_1}\lambda_2^2 + \frac{\partial W}{\partial I_2'}\lambda_2^2(\lambda_3^2+\lambda_1^2)+s \qquad (\lambda_{1,2,3} = 1+\varepsilon_{1,2,3})\\ \sigma_3 &= \frac{\partial W}{\partial I_1}\lambda_3^2 + \frac{\partial W}{\partial I_2'}\lambda_3^2(\lambda_1^2+\lambda_2^2)+s\end{aligned} \tag{1.1}$$

Here $\lambda_{1,2,3}$ are the measures of elongation, and $\varepsilon_{1,2,3}$ the linear strains on whose magnitude no constraints are imposed; I_1 and I_2' are the first and the reduced second invariant of the strain

$$I_1 = {}^1/_2\,(\lambda_1^2+\lambda_2^2+\lambda_3^2-3),\quad I_2' = 2\,(I_1 - I_2) = {}^1/_2\,(\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2-3) \tag{1.2}$$

Here s is an arbitrary constant; its indefiniteness is due to the assumption of invariance of the volume of the rubber during deformation; W is the strain potential energy of the rubber per unit volume.

The function $W(I_1, I_2')$ is determined by testing the given kind of rubber in several kinds of states of stress.

The most realistic is the four-term approximation of the function W proposed by Biderman [5]:

$$W = C_1 I_1 + C_2 I_2' - C_3 I_1^2 + C_4 I_1^3 \tag{1.3}$$

Here C_1, C_2, C_3, C_4 are the elastic constants of rubber, which have the dimensionality of a stress. The elastic modulus of rubber is expressed in terms of these constants, as follows:

$$E = 3\,(C_1 + C_2)$$

The law of rubber deformation is successfully represented quite accurately right down to rupture by using the function (1.3).

The state of stress of a spherical shell is biaxial, and $\sigma_3 = 0$. This affords a possibility of determining s.

After substitution of W from (1.3), we obtain

$$\begin{aligned}\sigma_1 &= (C_1 + C_2\lambda_2^2 - 2C_3I_1 + 3C_4I_1^2)\ (\lambda_1^2 - \lambda_3^2)\\ \sigma_2 &= (C_1 + C_2\lambda_1^2 - 2C_3I_1 + 3C_4I_1^2)\ (\lambda_2^2 - \lambda_3^2)\end{aligned} \qquad (1.4)$$

Let σ_0 and ε_0 denote the stress and strain existing in the shell in the precritical state. Upon passing to a new equilibrium mode these quantities undergo small changes. The principal stresses and strains will then be the following:

$$\sigma_1 = \sigma_0 + \Delta\sigma_1, \quad \varepsilon_1 = \varepsilon_0 + \Delta\varepsilon_1, \quad \sigma_2 = \sigma_0 + \Delta\sigma_2, \quad \varepsilon_2 = \varepsilon_0 + \Delta\varepsilon_2 \qquad (1.5)$$

The measures of elongation change correspondingly

$$\lambda_1 = 1 + \varepsilon_0 + \Delta\varepsilon_1, \qquad \lambda_2 = 1 + \varepsilon_0 + \Delta\varepsilon_2$$

Let us use the notation

$$1 + \varepsilon_0 = \alpha \qquad (1.6)$$

The quantity λ_3 is determined from the condition of invariance of the volume

$$\lambda_1\lambda_2\lambda_3 - 1 = 0 \qquad \text{for} \qquad \lambda_3 = \frac{1}{(\alpha + \Delta\varepsilon_1)(\alpha + \Delta\varepsilon_2)}$$

We obtain because of the smallness of $\Delta\varepsilon_1$ and $\Delta\varepsilon_2$

$$\lambda_1 = \alpha + \Delta\varepsilon_1, \qquad \lambda_2 = \alpha + \Delta\varepsilon_2, \qquad \lambda_3 = \frac{1}{\alpha^2}\left(1 - \frac{\Delta\varepsilon_1 + \Delta\varepsilon_2}{\alpha}\right) \qquad (1.7)$$

Substituting (1.7) and (1.5) into (1.2) and (1.4), we find

$$\sigma_0 = \left[C_1 + C_2\alpha^2 - C_3\left(2\alpha^2 + \frac{1}{\alpha^4} - 3\right) + \frac{3}{4}C_4\left(2\alpha^2 + \frac{1}{\alpha^4} - 3\right)^2\right]\left(\alpha^2 - \frac{1}{\alpha^4}\right) \qquad (1.8)$$

$$\Delta\sigma_1 + \Delta\sigma_2 = A\,(\Delta\varepsilon_1 + \Delta\varepsilon_2), \qquad \Delta\sigma_1 - \Delta\sigma_2 = B\,(\Delta\varepsilon_1 - \Delta\varepsilon_2) \qquad (1.9)$$

$$A = 2C_1\left(\alpha + \frac{2}{\alpha^5}\right) + 2C_2\alpha\left(2\alpha^2 + \frac{1}{\alpha^4}\right) - 2C_3\left[\left(\alpha + \frac{2}{\alpha^5}\right)\left(2\alpha^2 + \frac{1}{\alpha^4} - 3\right) + \right.$$

$$\left. + 2\alpha\left(\alpha - \frac{1}{\alpha^5}\right)^2\right] + 3C_4\left(2\alpha^2 + \frac{1}{\alpha^4} - 3\right)\left[\frac{1}{2}\left(2\alpha^2 + \frac{1}{\alpha^4} - 3\right)\left(\alpha + \frac{2}{\alpha^5}\right) + 2\alpha\left(\alpha - \frac{1}{\alpha^5}\right)^2\right]$$

$$B = 2\alpha\left[C_1 + C_2\frac{1}{\alpha^4} - C_3\left(2\alpha^2 + \frac{1}{\alpha^4} - 3\right) + \frac{3}{4}C_4\left(2\alpha^2 + \frac{1}{\alpha^4} - 3\right)^2\right] \qquad (1.10)$$

These quantities play the part of 'variable constant' of elasticity dependent on α.

2. Let us form the geometric relationship and assume that the shell remains a body of revolution during deviation from the spherical shape (Fig. 1).

After deviation, the meridian arc segment $AB = Rd\theta$ takes the position $A'B' = (1 + \varepsilon_0 + \Delta\varepsilon_1)\,Rd\theta$. The displacement along the normal will be $w_0 + w$, where w_0 is the subcritical displacement, commensurate with the radius and independent of the angle θ, and $w = w(\theta)$ is a small quantity. The linear u, and the angular ϑ displacements are just as small.

The relative elongations in the meridian and circumferential directions are

$$\begin{aligned}\varepsilon_1 &= \varepsilon_0 + \Delta\varepsilon_1 = \frac{w_0}{R} + \frac{w + u'}{R}\\ \varepsilon_2 &= \varepsilon_0 + \Delta\varepsilon_2 = \frac{w_0}{R} + \frac{w + u\operatorname{ctg}\theta}{R}\end{aligned} \qquad (2.1)$$

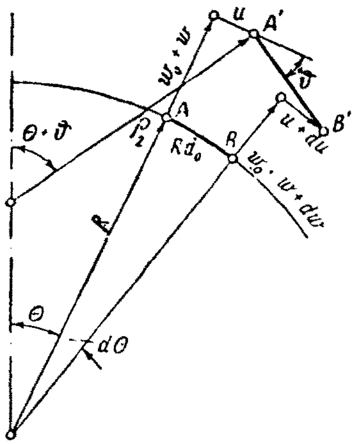

Fig. 1

Furthermore, we find the principal curvatures and we linearize the obtained expressions according to the condition of small w:

$$\frac{1}{\rho_2}=\frac{1}{R\alpha}-\frac{w+w'\operatorname{ctg}\theta}{R^2\alpha^2}, \qquad \frac{1}{\rho_1}=\frac{1}{R\alpha}-\frac{w''+w}{R^2\alpha^2} \tag{2.2}$$

The current thickness of the shell h^* equals $h\lambda_3$. According to (1.7), we obtain

$$\frac{1}{h^*}=\frac{\alpha^2}{h}\left(1+\frac{\Delta\varepsilon_1+\Delta\varepsilon_2}{\alpha}\right) \tag{2.3}$$

3. Let us form the equilibrium equations

$$\frac{\sigma_0+\Delta\sigma_1}{\rho_1}+\frac{\sigma_0+\Delta\sigma_2}{\rho_2}=\frac{p}{h^*}, \qquad \frac{\sigma_0+\Delta\sigma_1}{\rho_2}=\frac{p}{2h^*} \tag{3.1}$$

After substituting ρ_1, ρ_2, h^* and then linearizing, we find

$$\frac{2\sigma_0}{\alpha^3}=\frac{pR}{h} \tag{3.2}$$

$$\Delta\sigma_1+\Delta\sigma_2=\frac{\sigma_0}{R\alpha}(2w+w'\operatorname{ctg}\theta+w'')+\frac{pR}{h}\alpha^2(\Delta\varepsilon_1+\Delta\varepsilon_2)$$

$$2\Delta\sigma_1=2\frac{\sigma_0}{R\alpha}(w+w'\operatorname{ctg}\theta)+\frac{pR}{h}\alpha^2(\Delta\varepsilon_1+\Delta\varepsilon_2)$$

With the aid of (3.2) we eliminate the pressure in the last two espressions, and rewrite them as

$$\Delta\sigma_1+\Delta\sigma_2=\frac{\sigma_0}{R\alpha}(2w+w'\operatorname{ctg}\theta+w'')+\frac{2\sigma_0}{\alpha}(\Delta\varepsilon_1+\Delta\varepsilon_2)$$

$$\Delta\sigma_1-\Delta\sigma_2=\frac{\sigma_0}{R\alpha}(w'\operatorname{ctg}\theta-w'')$$

According to (1.9) and (2.1), we eliminate $\Delta\sigma_1$, $\Delta\sigma_2$, $\Delta\varepsilon_1$ and $\Delta\varepsilon_2$. We then obtain

$$(a-2)(2w+u'+u\operatorname{ctg}\theta)=2w+w'\operatorname{ctg}\theta+w''$$

$$b(u'-u\operatorname{ctg}\theta)=w'\operatorname{ctg}\theta-w'' \tag{3.3}$$

Here

$$a=A\alpha/\sigma_0, \qquad b=B\alpha/\sigma_0 \tag{3.4}$$

From the second Eq. of (3.3), it follows that

$$u=-\frac{1}{b}w'+C\sin\theta \tag{3.5}$$

and the first becomes

$$w''+w'\operatorname{ctg}\theta+n(n+1)w=-K\cos\theta \tag{3.6}$$

$$n(n+1)=\frac{2b(3-a)}{a+b-2}, \qquad K=2C\frac{b(a-2)}{a+b-2} \tag{3.7}$$

4. Eq. (3.6) is the Legendre equation written in canonical form with a right side.

Since there are no singularities at the poles, the quantity n should take on integer values $n = 0, 1, 2, \ldots$

In order to satisfy this condition, it is necessary to select an appropriate degree of elongation $\alpha = 1 + \varepsilon_0$.

Let us consider equilibrium modes corresponding to different values of n.

For $n = 0$ we obtain $a = 3$, $K = 2Cb / (b + 1)$. Eqs. (3.6) and (3.5) yield

$$w = D - \frac{Cb}{b+1}\cos\theta, \qquad u = \frac{Cb}{b+1}\sin\theta$$

We exclude the function ln tan ½θ from the considerations since it does not satisfy the condition of boundedness of w.

Let us agree to consider the equatorial plane as the fixed plane, then $C = 0$,

$$w = D, \qquad u = 0$$

Therefore, at some pressure there exists some other, slightly different but nevertheless also spherical mode in addition to the fundamental spherical mode. As is easily surmised, such a state corresponds to extremal values of the pressure p_1 or p_2 (Fig. 2), if such exist.

Indeed, for $n = 0$ the quantity $a = 3$, and therefore , $A = 3\sigma_0 / \alpha$. On the other hand, by determining the extremum of p from (3.2), we obtain

$$\frac{d\sigma_0}{d\alpha} = \frac{3\sigma_0}{\alpha}$$

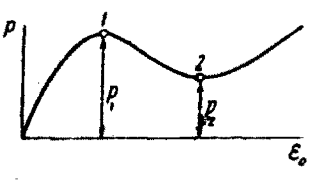

Fig. 2

Comparing the expressions (1.8) for σ_0 and (1.10) for A, we see that

$$\frac{d\sigma_0}{d\alpha} = A$$

and therefore, the condition $n = 0$ and the condition will be identical.

Let us note that for $n = 0$, all the relationships derived above are suitable, despite the assumption of small w, even for the description of the post-critical behavior of the shell since the latter retains its spherical shape.

For $n = 1$ we have $a = 2$, $K = 0$. Then from (3.6) and (3.5)

$$w = D\cos\theta, \qquad u = \frac{D}{b}\sin\theta + C\sin\theta$$

Again assuming the plane of the equator to be fixed, we obtain $w = D \cos\theta$, $u = 0$. The shell thickness h^* becomes variable $h^* = h\lambda_3$. According to (1.7) and (2.1)

$$h^* = \frac{h}{\alpha^3}\left(1 - \frac{2D}{R\alpha}\cos\theta\right)$$

Therefore, for $D > 0$ the shell thickness decreases in the upper hemisphere, and increases in the lower hemisphere (Fig. 3).

For subsequent n we have

$$b = \frac{n(n+1)(a-2)}{2(3-a) - n(n+1)}, \qquad K = Cn(n+1)\frac{a-2}{3-a}, \qquad w = DP_n(\theta) + \frac{K}{n(n+1)-2}\cos\theta$$

$$u = -\frac{D}{b}P_n'(\theta) + \frac{1}{b}\frac{K}{n(n+1)-2}\sin\theta + C\sin\theta$$

where $P_n(\theta)$ is the Legendre function of the first kind.

As in the two previous cases, we assume the plane of the equator is not displaced. Then

$$w = D\left[P_n(\theta) + P_n'\left(\frac{\pi}{2}\right)\frac{2(3-a) - n(n+1)}{3(3-a) - n(n+1)}\cos\theta\right]$$

$$u = D\,\frac{2(3-a) - n(n+1)}{n(n+1)(a-2)}\left[P_n'\left(\frac{\pi}{2}\right)\sin\theta - P_n'(\theta)\right]$$

For $n = 2$

$$P_n(\theta) = \frac{3\cos 2\theta + 1}{4}, \qquad w = \frac{D(3\cos 2\theta + 1)}{4}, \qquad u = D\,\frac{\sin 2\theta}{2(a-2)}$$

For $n = 3$

$$P_n(\theta) = {}^1/_8\,(5\cos 3\theta + 3\sin\theta)$$

$$w = \frac{1}{8}D\left(5\cos 3\theta + \frac{11a + 27}{a+1}\cos\theta\right), \qquad u = -D\,\frac{5}{16}\,\frac{a+3}{a-2}\,(\sin 3\theta + \sin\theta)$$

The respective equilibrium modes are shown in Fig. 3.

5. The question in to which of the found equilibrium modes are stable and which are not, requires further analysis. However, intuitive representations predict that the spherical 'zero' mode ($n = 0$), which may later pass into the first mode ($n = 1$), is stable in the initial stage of the loading. Higher equilibrium modes are apparently unstable, and not realized in practice if, moreover, special conditions are not produced.

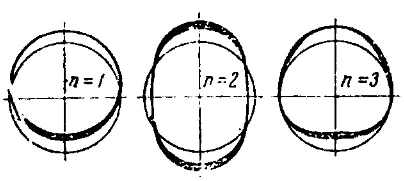

Fig. 3

An estimation of the stability of the spherical equilibrium mode by means of the maximum in the pressure, as is done in [1 and 2], cannot be considered without regard for the connection with loading conditions.

If the internal cavity of the sphere is connected with a large gas capacity, and the pressure is therefore independent of the shell deformation, then a jump increase in volume with subsequent rupture of the shell or passage to a new ascending branch of the $p = f(\varepsilon)$ curve will occur on reaching the extremal point (Fig. 2). Conservation of the spherical stable equilibrium mode is also possible here foe some kinds of tension diagrams.

If the gas is delivered to the internal cavity in small portions, as is customary, then passage through the maximum pressure does not involve qualitative changes in the loading process, and should not be considered as an indication of buckling.

The criteria obtained above for the passage of the pressure through the maximum ($n = 0$), and the passage to the mode with unilaterial thinning ($n = 1$) admit of a relatively simple geometric interpretation.

Let us assume there is a uniform biaxial strain diagram for rubber $\sigma_1 = \sigma_2 = \sigma_0$, $\sigma_3 = 0$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_0$ (Fig. 4). For the known rubber constant C_1, C_2, C_3 and C_4 this diagram may be constructed on the basis of (1.8), or be obtained directly from experiment, as is described in [5].

For $n = 0$ ($a = 3$), the passage of the pressure P throughout the extremum occurs while for $n = 1$ ($a = 2$), the equilibrium mode with unitaletral thinning exists in the neighborhood of the spherical mode.

Since $A = d\sigma_0 / da$, then according to (3.4)

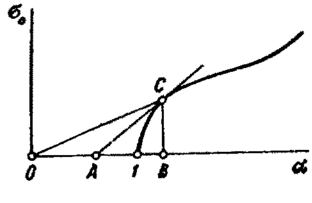

Fig. 4

$$a = \frac{d\sigma_0}{d\alpha}\frac{\alpha}{\sigma_0}$$

Hence, in conformity with the diagram (Fig. 4), the quantity a is determined by the ratio between the segments OB and AB, i.e., $a = OB/AB$.

As the shell elongates, this ratio diminishes, starting from infinity. The condition $a = 3$ (if it is possible) is satisfied for a lesser deformation than the condition $a = 2$. This means that the passage of the pressure through the maximum precedes the existence of the mode $n = 1$.

Depending on the kind of curve, it can be that neither condition will be satisfied, or condition $a = 3$ will be satisfied (even perhaps twice), and the condition $a = 2$ is not satisfied, i.e., the $n = 1$ mode does not exist. In particular, it can be seen after simple computations that this will hold for the two term Mooney approximation [6] for the function $W = C_1 I_1 + C_2 I_2'$, i.e., for $C_3 = C_4 = 0$.

If the $\sigma_0 = f(\varepsilon_0)$ diagram were linear, the pressure would have a maximum at $\alpha = 3/2$, i.e., for a one and one-half-fold increase in the sphere's diameter. The passage to the $n = 1$ mode would occur for a twofold increase in the diameter, i.e., for $\alpha = 2$.

If the critical elongation is known, then the pressure is determined by means of (3.2).

Conditions for the existence of modes with $n > 1$ are determined not only by the quantity a, but also by b, and they have not been given any successful geometric interpretation. In this case, we should turn to a numerical determination of n by means (1.8), (1.10), (3.4) and (3.7).

For natural rubber with eight parts sulfur by weight [4] the constants are

$$C_1 = 3.8\,\text{kg/cm}^2,\; C_2 = 0.2\,\text{kg/cm}^2,\; C_3 = 0.076\,\text{kg/cm}^2,\; C_4 = 3.68\cdot 10^{-3}\,\text{kg/cm}^2$$

or

$$\frac{C_2}{C_1} = 0.0525, \quad \frac{C_3}{C_1} = 0.02, \quad \frac{C_4}{C_1} = 0.00097$$

Performing the computations, we see that in this case a reaches the minimum value $a = 2.3$ after having diminished from infinity, and then again increases.. This means that for a shell fabricated from this rubber, only a spherical equilibrium mode is possible. The pressure extremum occurs at $a = 3$ or

$$p_{\max}^* = \frac{p_{\max}R}{C_1 h} = 1.338, \quad \alpha = 1.399; \quad p_{\min}^* = \frac{p_{\min}R}{C_1 h} = 1.218, \quad \alpha = 3.297$$

A small change in the relationships between the constants changes the picture. Thus for

$$C_2/C_1 = 0.02, \; C_3/C_1 = 0.03, \; C_4/C_1 = 0.001$$

we obtain four critical points

$$(n = 0,\ \alpha = 1.36,\ p^* = 1.25) \qquad (n = 1,\ \alpha = 1.79,\ p^* = 1.05)$$
$$(n = 1,\ \alpha = 2.84,\ p^* = 0.63) \qquad (n = 0,\ \alpha = 3.75,\ p^* = 0.53)$$

It can happen that all the critical points characterized by the number n from zero to infinity are located on the descending portion of the $p = f(\alpha)$ curve. Shown in Fig. 5 are appropriate curves for

$$\frac{C_2}{C_1} = 0.02, \qquad \frac{C_3}{C_1} = 0.04, \qquad \frac{C_4}{C_1} = 0.0005$$

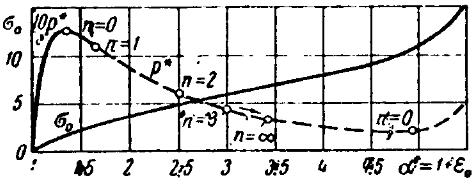

Fig. 5

The $p = f(\alpha)$ curve is shown dashed beyond the point $n = 1$ since the post-critical behavior of the system is unknown here.

A further analysis of examples is not meaningful since the elastic constants change rather strongly and are known only for some kinds of rubber.

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Translated by M.D.F.